STAT 302 2023/24 Winter Term 2 **Instructor: Vivian Meng**
Module 6 worksheet 1

New concepts

Continuous RV; uniform distribution; expected values; function of continous RV

Thought experiment

Imagine the following setup: you take a perfect string of length l, put a coordinate system along its length, with the origin at one end. Next, imagine having to cut the string at a *randomly selected* location, in a way that any location is **equally likely** to be chosen. Let the random variable X represent the location of the cut.

1. What do you think is the probability of the cut falling:

   in the first half of the string? ____; in the first quarter of the string? ____;

   at a location that has negative value? ____; at a location greater than l?____.

2. What is the general rule for the cut falling within a proportion a of its length, say within $[0, al]$, where $a \in (0, 1]$? $P(0 \leq X \leq al) =$ ____ .

3. In other words, prob. of the cut falling at a location between $-\infty$ to c (inclusive) is:

$$F(c) = P(X \leq c) = \begin{cases} \quad & , c < 0 \\ \quad & , 0 \leq c < l \\ \quad & , c \geq l \end{cases}$$

(a) Plot $F(a)$

(b) Plot $F'(a)$ where the derivative exists

4. How does $P(X \in [0, 1/2])$ compare to $P(X \in [0, 1/2))$? ____________.

Continuous uniform distribution over $[a, b]$

A RV X with continuous uniform distribution has density function that is uniform (i.e. non-zero constant) over the interval $[a, b]$ on the real line and 0 everywhere else. We say $X \sim \mathcal{U}(a, b)$.

Q1. Find $f(x)$, the probability density function (pdf) of a $\mathcal{U}(a, b)$ RV.

Given a density function, we can find the corresponding CDF because ____________________.

Q2. Find $F(x)$, the CDF of a $\mathcal{U}(a, b)$ RV.

Q3. Suppose $X \sim \mathcal{U}(0, 10)$. Find the following using the CDF

(a) $P(X > 3)$

(b) $P(2 \leq X < 12)$

(c) $P(X > 5 | X > 2)$

Expected values and probabilities of events

In general, for continuous RVs, calculation of expected values and probabilities of event $X \in B$, B being some interval on the real line, is similar to the calculation in the discrete case except that we replace summation with integration:

	Discrete	**Continuous**
$F(x)$	$\sum_{k \leq x} f(k)$	
$E[g(x)]$	$\sum_k g(k) f(k)$	
$P(X \in B)$	$\sum_{k \in B} f(k)$	

Q4. For $X \sim \mathcal{U}(a, b)$, calculate $E[X]$

For practice, try to calculate $E[X^2]$ and $Var(X)$ for a $\mathcal{U}(a, b)$ RV at home.

Transformations of a continuous RV

Let X be a RV with a known CDF $F_X(x)$ and $Y = g(X)$. We want to find CDF and pdf of Y.

• **When g is strictly monotonic**, the inverse function g^{-1} exists. This makes getting the CDF of Y, $F_Y(y)$ relatively easy.

step 1. Identify the range of Y. This informs us when $F_Y(y)$ will be 0 and when it will be 1.

step 2. Find $F_Y(y)$ by noting:
if g strictly monotonic increasing,

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) =$$

if g strictly monotonic decreasing,

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) =$$

step 3. Find pdf $f_y(y)$ by noting $f_y(y) = F'(y)$

• **But what if g not strictly monotonic?** Then we need to think about the shape of the function, maybe consider separating the domain of $g(\cdot)$ into regions where the function is strictly monotonic within each region...

Q5. Let $X \sim \mathcal{U}(-1, 1)$ Find the cdf and pdf of $Y = X^2$